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1986 J. Phys. A: Math. Gen. 19 479

(http://iopscience.iop.org/0305-4470/19/4/009)

J. Phys. A: Math. Gen. 19 (1986) 479-484. Printed in Great Britain

The Painlevé test, Bäcklund transformation and solutions of the reduced Maxwell–Bloch equations

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Received 2 January 1985, in final form 26 June 1985

Abstract. The Painlevé property for partial differential equations (PDEs) proposed by Weiss *et al* is studied for a system of PDEs, namely the reduced Maxwell-Bloch (RMB) equations. The RMB equations describe the propagation of short optical pulses through dielectric materials with a resonant non-degenerate transition. We demonstrate that the RMB system passes the Painlevé test, and we construct a Bäcklund transformation and solutions of the RMB equations.

1. Introduction

It is known that the Maxwell-Bloch (MB) equations describe the propagation of short ($<10^{-9}$ s) optical pulses in resonant media. The associated reduced Maxwell-Bloch (RMB) equations

$$\begin{aligned} u_x &= -\mu v, \\ v_x &= Ew + \mu u, \\ w_x &= -Ev, \\ E_t &= -v, \end{aligned} \tag{1.1}$$

are used to describe phenomena in non-linear optics, namely the theory of optical self-induced transparency. These equations were first derived by Eilbeck *et al* (1973). They are solved by the inverse scattering method by Gibbon *et al* (1973) and by Hirota's method in some detail by Caudrey *et al* (1974).

In this paper we demonstrate that the set of non-linear equations (1.1) passes the Painlevé test. Moreover, we derive a Bäcklund transformation and we give a method to construct solutions of the RMB equations.

The paper is organised as follows. Section 2 is devoted to the Painlevé property. In § 3 we give an application of the Painlevé property to a system of PDEs, namely the RMB equations, and finally we present a discussion of the results and some conclusions in § 4.

2. Integrability and the Painlevé test

Let us review some facts about ODEs. If an ODE does not have movable singularities, then the ODE is said to possess the Painlevé property. In other words, if all movable singularities are simple poles or non-movable critical points then the PDE possesses the Painlevé property. It is known that an ODE which possesses the Painlevé property is integrable (Tabor and Weiss 1981, Chang *et al* 1982, Bountis *et al* 1982). An ODE is said to be of Painlevé type if all its solutions possess the Painlevé property.

Consider now PDEs. Let m be the number of independent variables. Assume that the PDE has coefficients which are analytic on C^m. The Painlevé property is defined as follows (Ward 1984): if S is an analytic non-characteristic complex hypersurface in C^m, then every solution of the PDE which is analytic on $C^m \backslash S$ is meromorphic on C^m.

A weaker form of the Painlevé property was proposed by Weiss *et al* (1983). They looked for solutions of the PDE in the form

$$u = \phi^n \sum_{j=0}^{\infty} u_j \phi^j, \tag{2.1}$$

where ϕ is an analytic function whose vanishing defines a non-characteristic hypersurface S. The motivation of the ansatz (2.1) comes from the theory of ODEs.

Inserting the expansion (2.1) into the PDE leads to conditions on n and recursion relations for the functions u_j which are certain analytic functions of the independent variables. The Painlevé property here states that n should be an integer, that the recursion relations should be consistent and that the series expansion (2.1) should contain the correct number of arbitrary functions. Resonances are those values of j at which it is possible to introduce arbitrary functions into the expansion. Notice that it may happen that more than one branch arises. An example is the wave equation $u_{tt} - u_{xx} = \exp(u) + \exp(-2u)$. The expansion could, *a priori*, miss some essential singularities. This behaviour is well known for ODEs. If we study the case with more than one field, then the expansion is given by

$$u_k = \phi^{n_k} \sum_{j=0}^{\infty} u_{kj} \phi^j. \tag{2.2}$$

Various authors (Chudnovsky *et al* 1983, Grauel 1985a, b, Steeb *et al* 1983, 1984) have applied the weaker form (2.1) of the Painlevé property and we use the weaker form (2.2) to perform our Painlevé test.

3. Application to systems of partial differential equations

In the technique described by Weiss *et al* (1983) the quantities are considered in the complex domain. For the sake of simplicity we do not change our notation. The Painlevé test of PDEs can be performed in the same manner as the singular point analysis for ODEs (Ablowitz *et al* 1980). First of all we determine the dominant behaviour, i.e. we calculate the exponents n_k and expansion coefficients u_{k0} ($k = 1, \ldots, 4$). Inserting ($u_1 \equiv u$, $u_2 \equiv v$, $u_3 \equiv w$, $u_4 \equiv E$) the ansatz

$$u_k \sim \phi^{n_k} u_{k0} \tag{3.1}$$

into (1.1) shows that for certain values of n_k, two or more terms in the system of RMB equations may balance, and the rest can be ignored. We call such terms the leading terms. We find

$$n_1 = -1, \qquad n_2 = -2, \qquad n_3 = -2, \qquad n_4 = -1. \tag{3.2}$$

There is only one branch. The expansion coefficients u_0, v_0, w_0, E_0 are determined by

$$\begin{aligned}
&-u_0\phi_x+\mu v_0=0,\\
&2v_0\phi_x+E_0w_0=0,\\
&-2w_0\phi_x+E_0v_0=0,\\
&E_0\phi_t+v_0=0.
\end{aligned}\tag{3.3}$$

From (3.3) it follows that

$$\begin{aligned}
&u_0=2i\mu\phi_t,\\
&v_0=2i\phi_x\phi_t,\\
&w_0=-2\phi_x\phi_t,\\
&E_0=2i\phi_x.
\end{aligned}\tag{3.4}$$

In the next step we determine the resonances. To do so we introduce (2.2) with (3.2) into (1.1) and obtain

$$\sum_{j=0}^{\infty}[u_{jx}\phi^{j-1}+u_j(j-1)\phi^{j-2}\phi_x]+\mu\sum_{j=0}^{\infty}v_j\phi^{j-2}=0,\tag{3.5a}$$

$$\sum_{j=0}^{\infty}[v_{jx}\phi^{j-2}+v_j(j-2)\phi^{j-3}\phi_x]-\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}E_jw_k\phi^{j+k-3}-\mu\sum_{j=0}^{\infty}u_j\phi^{j-1}=0,\tag{3.5b}$$

$$\sum_{j=0}^{\infty}[w_{jx}\phi^{j-2}+w_j(j-2)\phi^{j-3}\phi_x]+\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}E_jv_k\phi^{j+k-3}=0,\tag{3.5c}$$

$$\sum_{j=0}^{\infty}[E_{jt}\phi^{j-1}+E_j(j-1)\phi^{j-2}\phi_t]+\sum_{j=0}^{\infty}v_j\phi^{j-2}=0.\tag{3.5d}$$

The resonances r are determined from the coefficients with the factors ϕ^{j-2} in (3.5a) and (3.5d), ϕ^{j+k-3} in (3.5b) and (3.5c).

For the coefficients with the factors ϕ^{j-2} and ϕ^{j-3} we have to put $j=r$ and for the coefficients with the factors ϕ^{j+k-3} we have to put $j=r$, $k=0$ and $j=0$, $k=r$. From (3.5) it follows that

$$\begin{pmatrix}(r-1)\phi_x & \mu v_r & 0 & 0\\ 0 & (r-2)\phi_x & -E_0 & -w_0\\ 0 & E_0 & (r-2)\phi_x & v_0\\ 0 & 1 & 0 & (r-1)\phi_x\end{pmatrix}\begin{pmatrix}u_r\\ v_r\\ w_r\\ E_r\end{pmatrix}=0.\tag{3.6}$$

The resonances are determined from the condition that the determinant of the matrix on the left-hand side of (3.6) is equal to zero. Taking into account (3.4), we obtain

$$r_1=-1,\qquad r_2=1,\qquad r_3=2,\qquad r_4=4.\tag{3.7}$$

The resonance at $r_1=-1$ corresponds to the arbitrary (undefined) singularity manifold $\phi=0$.

From (3.5) we obtain (3.4) and

$$\begin{pmatrix}0 & 1 & 0 & 0\\ 0 & -\phi_x & -2i\phi_x & 2\phi_x\phi_t\\ 0 & 2i\phi_x & -\phi_x & 2i\phi_x\phi_t\\ 0 & 1 & 0 & 0\end{pmatrix}\begin{pmatrix}u_1\\ v_1\\ w_1\\ E_1\end{pmatrix}=\begin{pmatrix}-2i\phi_{xt}\\ -2i\phi_{xx}\phi_t-2i\phi_x\phi_{tx}\\ 2\phi_{xx}\phi_t+2\phi_x\phi_{tx}\\ -2i\phi_{xt}\end{pmatrix}.\tag{3.8}$$

This equation arises at the resonance $r_2=1$. The solution of (3.8) is given by

$$\begin{aligned} v_1 &= -2\mathrm{i}\phi_{xt}, \\ w_1 &= 2\phi_{xt}, \\ E_1 &= -\mathrm{i}\phi_x^{-1}\phi_{xx}, \end{aligned} \tag{3.9}$$

and u_1 can be chosen arbitrarily. The equations for the expansion coefficients u_m, v_m, w_m, E_m are determined by the recursion relations

$$u_{(m-1)x}+(m-1)u_m\phi_x+\mu v_m=0, \tag{3.10a}$$

$$v_{(m-1)x}-(2-m)v_m\phi_x-\sum_{j=0}^{m} E_j w_{(m-j)}-\mu u_{(m-2)}=0, \tag{3.10b}$$

$$w_{(m-1)x}-(2-m)w_m\phi_x-\sum_{j=0}^{m} E_j v_{(m-j)}=0, \tag{3.10c}$$

$$E_{(m-1)t}+(m-1)E_m\phi_t+v_m=0, \tag{3.10d}$$

where $m \geqslant 1$ for (3.10a) and (3.10d) and $m \geqslant 2$ for (3.10b) and (3.10c) hold. At the resonance $r_3=2$ we find that two of the four resulting equations are identical. We find that v_2 or E_2 can be chosen arbitrarily. If we choose E_2 as the arbitrary function, then it follows that

$$\begin{aligned} v_2 &= -\phi_t E_2-\mathrm{i}\phi_x^{-2}\phi_{xx}\phi_{xt}+\mathrm{i}\phi_x^{-1}\phi_{xxt}, \\ u_2 &= -\mu\phi_x^{-1}v_2-\phi_x^{-1}u_{1x}, \\ w_2 &= \mathrm{i}v_2-\mu^2\phi_x^{-1}\phi_t. \end{aligned} \tag{3.11}$$

The four fields u_3, v_3, w_3, E_3 are determined by

$$\begin{pmatrix} 2\phi_x & \mu & 0 & 0 \\ 0 & \phi_x & -2\mathrm{i}\phi_x & 2\phi_x\phi_t \\ 0 & 2\mathrm{i}\phi_x & \phi_x & 2\mathrm{i}\phi_x\phi_t \\ 0 & 1 & 0 & 2\phi_t \end{pmatrix} \begin{pmatrix} u_3 \\ v_3 \\ w_3 \\ E_3 \end{pmatrix} = \begin{pmatrix} -u_{2x} \\ -v_{2x}+E_2w_1+E_1w_2+\mu u_1 \\ -w_{2x}-E_2v_1-E_1v_2 \\ -E_{2t} \end{pmatrix}. \tag{3.12}$$

At the resonance $r_4=4$ and we have the following set of equations:

$$\begin{pmatrix} 3\phi_x & \mu & 0 & 0 \\ 0 & 2\phi_x & -E_0 & -w_0 \\ 0 & E_0 & 2\phi_x & v_0 \\ 0 & 1 & 0 & 3\phi_t \end{pmatrix} \begin{pmatrix} u_4 \\ v_4 \\ w_4 \\ E_4 \end{pmatrix} = \begin{pmatrix} -u_{3x} \\ -v_{3x}+E_3w_1+E_1w_3+E_2w_2+\mu u_2 \\ -w_{3x}-E_3v_1-E_1v_3-E_2v_2 \\ -E_{3t} \end{pmatrix}. \tag{3.13a, b, c, d}$$

The equations (3.13b) and (3.13c) are equal and therefore three equations remain for the fields u_4, v_4, w_4, E_4. Consequently (1.1) passes the Painlevé test, i.e. we have an expansion of the form (2.4) in which four 'expansion coefficients' can be chosen arbitrarily. By a cut-off of the series (2.4) we find a Bäcklund transform. Requiring

$$u_j=0, j\geqslant 2, \qquad v_j=0, j\geqslant 3, \qquad w_j=0, j\geqslant 3, \qquad E_j=0, j\geqslant 2 \tag{3.14}$$

yields

$$\begin{aligned} u &= 2\mathrm{i}\mu\phi_t\phi^{-1}+u_1, \\ v &= 2\mathrm{i}\phi_x\phi_t\phi^{-2}+v_1\phi^{-1}+v_2, \\ w &= -2\phi_x\phi_t\phi^{-2}+w_1\phi^{-1}+w_2, \\ E &= 2\mathrm{i}\phi_x\phi^{-1}+E_1, \end{aligned} \tag{3.15}$$

$$v_{1x}-E_0w_2-E_1w_1-\mu u_0=0, \qquad w_{1x}+E_0v_2+E_1v_1=0,$$

and

$$v_1 = -2\mathrm{i}\phi_{xt}, \qquad w_1 = 2\phi_{xt}, \qquad E_1 = -\mathrm{i}\phi_x^{-1}\phi_{xx}. \tag{3.16}$$

The functions u_1, v_2, w_2, E_1 satisfy (1.1).

The functions u_0, v_0, w_0, E_0 are now given by

$$\begin{aligned} &u_0 = 2\mathrm{i}\mu\phi_t, &&v_0 = 2\mathrm{i}\phi_x\phi_t, \\ &w_0 = \mathrm{i}v, &&E_0 = 2\mathrm{i}\phi_x. \end{aligned} \tag{3.17}$$

We can calculate the arbitrary function u_1. To do so we insert the functions u_0, E_0, v_1, w_1 and E_1 into (3.15). We obtain

$$\begin{aligned} v_2 &= -\mathrm{i}\phi_x^{-2}\phi_{xx}\phi_{xt} + \mathrm{i}\phi_x^{-1}\phi_{xxt}, \\ w_2 &= -\phi_{xxt}\phi_x^{-1} + \phi_x^{-2}\phi_{xx}\phi_{xt} - \mu^2\phi_x^{-1}\phi_t. \end{aligned} \tag{3.18}$$

If we insert v_2 and w_2 into (1.1) for the functions u_1, v_2, w_2 and E_1 then it follows that

$$u_1 = \mathrm{i}\mu\phi_x^{-1}\phi_{xt}. \tag{3.19}$$

Consequently

$$\begin{aligned} u &= 2\mathrm{i}\mu\phi_t\phi^{-1} - \mathrm{i}\mu\phi_x^{-1}\phi_{xt}, \\ v &= 2\mathrm{i}\phi_x\phi_t\phi^{-2} - 2\mathrm{i}\phi_{xt}\phi^{-1} - \mathrm{i}\phi_x^{-2}\phi_{xx}\phi_{xt} + \mathrm{i}\phi_x^{-1}\phi_{xxt}, \\ w &= \mathrm{i}v - \mu^2\phi_x^{-1}\phi_t, \qquad E = 2\mathrm{i}\phi_x\phi^{-1} - \mathrm{i}\phi_x^{-1}\phi_{xx}, \end{aligned} \tag{3.20}$$

and ϕ satisfies

$$\{\phi, x\}_t + \mu^2(\phi_t/\phi_x)_x = 0. \tag{3.21}$$

The symbol $\{\phi, x\}$ characterises the Schwarzian derivative (Hille 1976) which is given by

$$\{\phi, x\} = \phi_{xxx}\phi_x^{-1} - \tfrac{3}{2}(\phi_{xx}/\phi_x)^2. \tag{3.22}$$

Equation (3.22) is invariant under the Moebius group (Schwerdtfeger 1979). That is, if ϕ satisfies (3.22), then

$$\psi = (a\phi + b)/(c\phi + d), \qquad ad - bc \neq 0, \tag{3.23}$$

also satisfies (3.22). The Schwarzian derivative result and its significance in the PDE context were first pointed out by Weiss (1983).

We have expressed the fields u, v, w and E in terms of ϕ. The field is restricted by the condition (3.21). Therefore, we must find solutions of (3.21) to obtain solutions of the RMB equations (1.1) for the fields u, v, w and E in (3.20). Solutions can be found by Hirota's method (1976). A solution is given by

$$\phi = 1 + \exp(x + q_1(t) + c_1) \tag{3.24}$$

and for the electromagnetic wave

$$E = -(\mu \sin \eta)/(\cos h\zeta + \cos \eta), \tag{3.25}$$

where η and ζ are real quantities. If we take into account that η and ξ are real quantities, consequently we have $E = -\mu \operatorname{sech} \zeta$ and this special solution can be identified with a solution which is given by Bullough *et al* (1979). More about the Painlevé property and Hirota's method can be found in the recent paper of Radmore *et al* (1984).

4. Summary and conclusion

We have discussed the reduced Maxwell-Bloch equation and we found that the Painlevé property is a valuable analytical test for the integrability of systems of PDEs. We found that the reduced Maxwell-Bloch system in the sharp line limit possesses the Painlevé property and this means that the system of PDEs is integrable. Moreover, we found a Bäcklund transformation and it is found that the equation for the 'singular surface' can be expressed in terms of the Schwarzian derivative. This equation is invariant under the Moebius group. Finally we have given a solution for the field ϕ.

It is conjectured that if a non-linear PDE of a system possesses the Painlevé property, then we conclude that this equation is integrable. On the other hand we cannot conclude, in general, that a PDE which is integrable has the Painlevé property.

References

Ablowitz M J, Ramani A and Segur H 1980 *J. Math. Phys.* **21** 715
Bountis T, Segur H and Vivaldi F 1982 *Phys. Rev.* A **25** 1257
Bullough R K, Jack P M, Kitchenside and Saunders R 1979 *Phys. Scr.* **20** 364
Caudrey P J *et al* 1974 *J. Inst. Math. Appl.* **14** 375
Chang Y F, Tabor M and Weiss J 1982 *J. Math. Phys.* **23** 531
Chudnovsky D V, Chudnovsky G V and Tabor M 1983 *Phys. Lett.* **97A** 268
Eilbeck J C, Gibbon J D, Caudrey P J and Bullough R K 1973 *J. Phys. A: Math., Nucl. Gen.* **6** 1337
Gibbon J D, Caudrey P J, Bullough R K and Eilbeck J C 1973 *Lett. Nuovo Cimento* **8** 775
Grauel A 1985a *Solid State Commun.* **53** 765
—— 1985b *Physica* **132A** 557
Hille E 1976 *Ordinary Differential Equation in the Complex Domain* (London: Wiley Interscience)
Hirota R 1976 in *Bäcklund Transformations, the Inverse Scattering Method, Solitons, and Their Applications* ed R M Miura *Lecture Notes in Mathematics* **515** (Berlin: Springer)
Radmore D, Wood D, Gibbon J D and Tabor M 1984 *The Painlevé Property and Hirota's Method, preprint* Department of Applied Physics, Columbia University
Schwerdtfeger H 1979 *Geometry of Complex Numbers* (New York: Dover)
Steeb W H and Grauel A 1984 *J. Phys. Soc. Japan* **53** 1901
Steeb W H, Grauel A, Kloke M and Spieker B M 1984 *Phys. Scr.* **31** 558
Steeb W H, Kloke M, Spieker B M and Oevel W 1983 *J. Phys. A: Math. Gen.* **16** L447
Tabor M and Weiss J 1981 *Phys. Rev.* A **24** 2157
Ward R S 1984 *Phys. Lett.* **102A** 279
Weiss J 1983 *J. Math. Phys.* **24** 1405
Weiss J, Tabor M and Carnevale G 1983 *J. Math. Phys.* **24** 552